LECTURE 2: DELIGNE COHOMOLOGY AND L-FUNCTIONS

In this lecture we introduce Deligne cohomology $H^i_{\mathcal{D}}(X, \mathbb{R}(i))$. These vector spaces serve as the targets of our regulator maps. Unfortunately the definition will seem rather unmotivated. Rest assured that the origin of Deligne cohomology is well-understood, and the ad-hoc presentation given here is simply a by-product of needing to keep within the schedule.

Notation 0.1. Let Λ be a subring of $\mathbb{C}$. We shall write $\Lambda(n) := (2\pi i)^n \Lambda \subset \mathbb{C}$.

1. Deligne cohomology

Let X be a complex manifold. Consider the following complex of sheaves of abelian groups on X:

$$\Lambda(n)_{\mathcal{D}} := \Lambda(n) \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1}_X$$

in degrees $0, \dots, n-1$, where Ω^i_X is the sheaf of holomorphic differential i-forms on X.

There is a product map

$$\cup : \Lambda(n)_{\mathcal{D}} \otimes \Lambda(m)_{\mathcal{D}} \to \Lambda(n+m)_{\mathcal{D}}$$

defined by

$$x \cup y := \begin{cases} xy & \text{if } \deg(x) = 0 \\ x \wedge dy & \text{if } \deg(x) > 0 \text{ and } \deg(y) = m \\ 0 & \text{otherwise}. \end{cases}$$

Definition 1.1. The Deligne cohomology of X with Λ-coefficients is defined to be

$$H^i_{\mathcal{D}}(X, \Lambda(n)) := \mathbb{H}^i(X, \Lambda(n)_{\mathcal{D}})$$

for integers $i, n \geq 0$.

Some homological algebra 1.2. Recall that the hypercohomology of a complex of sheaves $C^\bullet$ on X is, by definition, the cohomology of an injective resolution $C^\bullet \to I^\bullet$ on X. That is,

$$\mathbb{H}^*(X, C^\bullet) := H^*(\Gamma(X, I^\bullet)).$$

In the exercise sheet I will give another way of describing hypercohomolgy which is sometimes more useful for computations.

The product map on complexes induces a product on cohomology:

$$\cup : H^i_{\mathcal{D}}(X, \Lambda(n)) \times H^j_{\mathcal{D}}(X, \Lambda(m)) \to H^{i+j}_{\mathcal{D}}(X, \Lambda(n+m)).$$

If X is a smooth projective variety over $\mathbb{C}$ then its set of complex points $X^{\mathrm{an}} = X(\mathbb{C})$ is a complex manifold, and we define

$$H^i_{\mathcal{D}}(X, \Lambda(n)) := H^i_{\mathcal{D}}(X^{\mathrm{an}}, \Lambda(n)_{\mathcal{D}}).$$

If X is a smooth projective variety over $\mathbb{R}$, let $X_{\mathbb{C}} := X \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$. Then we define

$$H^i_{\mathcal{D}}(X, \Lambda(n)) := H^i_{\mathcal{D}}(X_{\mathbb{C}}, \Lambda(n)_{\mathcal{D}})^+$$

where the superscript + means that we take the fixed elements under the endomorphism

$$F_\infty^* \circ (\text{complex conjugation on } \Lambda)$$

where F_∞ is the anti-holomorphic involution on $(X_\mathbb{C})^{\mathrm{an}}$ induced by complex conjugation (i.e. $X_\mathbb{C}^{\mathrm{an}}$ is cut out of $\mathbb{P}_\mathbb{C}^{n,\mathrm{an}}$ by some polynomials with $\mathbb{R}$-coefficients. The anti-holomorphic involution is the map induced by taking complex conjugates of the homogeneous coordinates).

For X is a smooth projective variety over $\mathbb{R}$, note that under the comparison isomorphism from de Rham to singular cohomology

$$H^*_{\mathrm{dR}}(X_\mathbb{C}/\mathbb{C}) \simeq H^*_{\mathrm{dR}}(X/\mathbb{R}) \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\sim} H^*_{\mathrm{sing}}(X_\mathbb{C}, \mathbb{C}) \simeq H^*_{\mathrm{sing}}(X_\mathbb{C}, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}$$

the involution $F_\infty^* \otimes (\text{complex conjugation on } \mathbb{C})$ on the right corresponds to the involution $\mathrm{id} \otimes (\text{complex conjugation on } \mathbb{C})$ on the left.

One can show that Deligne cohomology satisfies a list of properties analogous to the list given for motivic cohomology in Lecture 1 (localisation, homotopy invariance etc).

Some homological algebra 1.3. Recall that for a morphism of complexes $f : A^\bullet \to B^\bullet$, the cone or mapping fibre of f is the complex

$$\mathrm{MF}(f) := A^\bullet[1] \oplus B^\bullet$$

where the differential is

$$\begin{aligned} A^{n+1} \oplus B^n &\to A^{n+2} \oplus B^{n+1} \\ (a, b) &\mapsto (-da, f(a) + db) \end{aligned}$$

(where d means either the differential on $A^\bullet$ or on $B^\bullet$ depending on the situation). The point of a cone is that it sits in a long exact sequence

$$\ldots \longrightarrow \mathbb{H}^i(A^\bullet) \xrightarrow{\mathbb{H}^i(f)} \mathbb{H}^i(B^\bullet) \longrightarrow \mathbb{H}^i(MF(f)) \longrightarrow \mathbb{H}^{i+1}(A^\bullet) \xrightarrow{\mathbb{H}^{i+1}(f)} \ldots .$$

For $n \geq 0$ let $\Omega_X^{\geq n} = (0 \to \ldots \to 0 \to \Omega_X^n \to \Omega_X^n \to \cdots)$ be the truncated de Rham complex. Let

$$f : \Omega_X^{\geq n} \oplus \Lambda(n) \to \Omega_X^\bullet$$

be the morphism of complexes given by the difference of the inclusions. Then

$$\mathrm{MF}(f)[-1] \xrightarrow{\sim} \Lambda(n)_\mathcal{D} .$$

In particular, if X is a smooth complex variety over $\mathbb{R}$ or $\mathbb{C}$, we have a long exact sequence

$$\ldots \to H^i_\mathcal{D}(X, \mathbb{R}(n)) \to \mathrm{Fil}^n H^i_{\mathrm{dR}}(X) \oplus H^i_{\mathrm{sing}}(X, \mathbb{R}(n)) \to H^i_{\mathrm{dR}}(X) \to H^{i+1}_\mathcal{D}(X, \mathbb{R}(n)) \to \ldots .$$

Recall that $\mathrm{Fil}^n H^i_{\mathrm{dR}}(X)$ above is defined as the image of the map on cohomology induced by the inclusion $\Omega_X^{\geq n} \hookrightarrow \Omega_X^\bullet$ (the filtration $\mathrm{Fil}^\bullet H^i_{\mathrm{dR}}(X)$ is called the Hodge filtration). Note that de Rham cohomology and singular cohomology are finite dimensional vector spaces, therefore so is Deligne cohomology.

2. Examples

Let X be a complex variety.

(1) Clearly $\Lambda(0)_{\mathcal{D}} = \Lambda$, so

$$H^i_{\mathcal{D}}(X, \Lambda(0)) = H^i_{\text{sing}}(X, \Lambda)$$

is singular cohomology of X with Λ-coefficients.

(2) The exponential sequence

$$0 \to (2\pi i)\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0\,.$$

shows that the hypercohomology of $\mathbb{Z}(1)_{\mathcal{D}} = ((2\pi i)\mathbb{Z} \to \mathcal{O}_X)$ is isomorphic to the cohomology of $\mathcal{O}_X^*[-1]$ via the exponential. That is,

$$H^i_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^{i-1}(X, \mathcal{O}_X^*)\,.$$

As a special case, we get $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong \operatorname{Pic}(X)$.

(3) Consider the following morphism of complexes

$$\begin{array}{ccccc}
(2\pi i)^2\mathbb{Z} & \longrightarrow & \mathcal{O}_X & \xrightarrow{\;d\;} & \Omega^1_X \\
\downarrow & & \downarrow{\scriptstyle x \mapsto \exp(x/2\pi i)} & & \downarrow{\scriptstyle =} \\
0 & \longrightarrow & \mathcal{O}_X^* & \xrightarrow[x \mapsto dx/x]{d\log} & \Omega^1_X
\end{array}$$

Notice that the complex on the top row is $\mathbb{Z}(2)_{\mathcal{D}}$. The morphism of complexes induces an isomorphism on hypercohomology, i.e.

$$H^i_{\mathcal{D}}(X, \mathbb{Z}(2)) \cong \mathbb{H}^{i-1}(\mathcal{O}_X^* \xrightarrow{d\log} \Omega^1_X)\,.$$

(4) For $n > i+1$, the long exact sequence gives

$$H^{i+1}_{\mathcal{D}}(X, \mathbb{R}(n)) \cong H^i_{\text{sing}}(X, \mathbb{C}/\mathbb{R}(n)).$$

(5) Let $X = \operatorname{Spec}\mathbb{C}$. Then we get that

$$H^i_{\mathcal{D}}(\mathbb{C}, \mathbb{R}(n)) = \begin{cases} \mathbb{R} & \text{if } n = i = 0 \\ \mathbb{R}(n-1) & \text{if } i = 1 \text{ and } n \geq 1 \\ 0 & \text{otherwise}\,. \end{cases}$$

In particular, let F be a number field. Write $X = \operatorname{Spec}(F) \times \mathbb{C}$ and let $X_{\mathbb{R}}$ be its natural structure as a real variety. Then

$$H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) = \left(\prod_{\sigma \in \operatorname{Hom}(\mathrm{F}, \mathbb{C})} \mathbb{R}(n-1) \right)^+ \cong \begin{cases} \mathbb{R}(n-1)^{r_2} & \text{if } n \text{ is even} \\ \mathbb{R}(n-1)^{r_1 + r_2} & \text{if } n \text{ is odd}\,. \end{cases}$$

for $n \geq 1$, where r_1 (resp. r_2) denotes the number of real (resp. complex pairs) of embeddings of F.

3. The Beilinson regulator

For all $i, n \geq 0$, there is a map

$$r_{\mathcal{B}} : H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \to H^i_{\mathcal{D}}(X, \mathbb{R}(n)).$$

called the Beilinson regulator. Unfortunately, in order to actually define $r_{\mathcal{B}}$, one passes through the quasi-projective world. So one first needs to defines a generalised version of Deligne cohomology in the setting that X is smooth quasi-projective, called Deligne-Beilinson cohomology. Defining Deligne-Beilinson cohomology would require relying on too many prerequisites for us to get through in a reasonable amount of time, and even then the definition of $r_{\mathcal{B}}$ is an order of magnitude harder still. So, for time reasons and to keep the homological prerequisites in check, I'll only give an idea the definition and instead focus on some examples. Besides, we are interested in the the Beilinson conjectures (see the next lecture), which only concern the case where X is smooth and projective. For now let me just point out that we have seen one instance of the Beilinson regulator already:

Let F be a number field and let $X = \mathrm{Spec}\,(k) \times \mathbb{C}$. Then we have seen that

$$\begin{array}{ccc} H^1_{\mathcal{M}}(X, \mathbb{Q}(1)) & \xrightarrow{r_{\mathcal{B}}} & H^1_{\mathcal{D}}(X, \mathbb{R}(1)) \\ \downarrow \cong & & \cong \downarrow \\ \mathbb{C}^* \otimes \mathbb{Q} & \longrightarrow & \mathbb{R} \end{array}$$

Under these isomorphisms, the Beilinson regulator $r_{\mathcal{B}}$ corresponds to the map

$$\mathbb{C}^* \otimes \mathbb{Q} \to \mathbb{R}$$
$$f \otimes 1 \mapsto \log |f|.$$

In particular, if F is a number field of degree $[F : \mathbb{Q}] = d = r_1 + 2r_2$ the composition

$$F^* \otimes \mathbb{Q} \cong H^1_{\mathcal{M}}(F, \mathbb{Q}(1)) \xrightarrow{\text{base change}} H^1_{\mathcal{M}}(X_{\mathbb{R}}, \mathbb{R}(1)) \xrightarrow{r_{\mathcal{B}}} H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(1)) \simeq \mathbb{R}^{r_1+r_2}$$

is precisely the Dirichlet regulator. We will discuss

$$r_{\mathcal{B}} : H^1_{\mathcal{M}}(F, \mathbb{Q}(n)) \to H^1_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$$

for $n \geq 1$ in Lecture 4.

3.1. **Quick sketch of a construction.** Everything in this section can be ignored. Let X be a quasi-projective variety. Something called Jouanolou's trick says that there exists an affine variety $\mathrm{Spec}\, R$ with a map $\mathrm{Spec}\, R \to X$ making $\mathrm{Spec}\, R \to X$ into a vector bundle over X. Using localisation and $\mathbb{A}^1$-homotopy invariance, one reduces to constructing the regulator for $\mathrm{Spec}\, R$, where R is a finitely generated $\mathbb{R}$-algebra. There is a map

$$H^i_{\mathcal{M}}(R, \mathbb{Q}(n)) \to H_{2n-i}(\mathrm{GL}(R), \mathbb{R}) := \varinjlim_N H_{2n-i}(\mathrm{GL}_N(R), \mathbb{R})$$

(one way to see that there is such a map is by recalling the Grothendieck-Riemann-Roch formula from Lecture 1 to view $H^i_{\mathcal{M}}(R, \mathbb{Q}(n))$ as a summand of $K_{2n-i}(R) \otimes \mathbb{Q}$ and then knowing that there is a Hurewicz map $K_{2n-i}(R) \to H_{2n-i}(\mathrm{GL}(R))$ because algebraic K-theory is defined as a homotopy group of a space which is close to the classifying space $B\mathrm{GL}(R)$). Re-indexing, it suffices to construct a map

$$H_j(\mathrm{GL}(R), \mathbb{R}) \to H^{2m-j}_{\mathcal{D}}(R, \mathbb{R}(m)).$$

It is known that

$$H^{2n-1}_{\mathrm{sing}}(B_\bullet \mathrm{GL}_N(R)(\mathbb{C}), \mathbb{Q}) = 0$$

and

$$\bigoplus_{n\geq 0} H^{2n}_{\mathrm{sing}}(B_\bullet \mathrm{GL}_N(R)(\mathbb{C}), \mathbb{Q}(n)) = \mathbb{Q}[c_1, \dots, c_N]$$

where $c_n \in H^{2n}_{\mathrm{sing}}(B_\bullet \mathrm{GL}_N(R)(\mathbb{C}), \mathbb{Q}(n))$ is the n-th Chern class of the universal bundle $\mathscr{E}$ over the simplicial scheme $B_\bullet \mathrm{GL}_N(R)$. Now, pulling back along the evaluation map $\mathrm{Spec}\, R \times B_\bullet \mathrm{GL}_N(R) \to B_\bullet \mathrm{GL}_N$ gives a map for all p, q

$$\begin{aligned} \mathrm{ev}^* : H^p_{\mathcal{D}}(B_\bullet \mathrm{GL}_N, \mathbb{R}(q)) \to & H^p_{\mathcal{D}}(\mathrm{Spec}\, R \times B_\bullet \mathrm{GL}_N(R), \mathbb{R}(q)) \\ & \cong \bigoplus_j H^{p-j}_{\mathcal{D}}(\mathrm{Spec}\, R, \mathbb{R}(q)) \otimes H^j_{\mathrm{sing}}(B_\bullet \mathrm{GL}_N(R), \mathbb{R}) \end{aligned}$$

where the isomorphism is the Künneth formula for Deligne-Beilinson cohomology. Then there is a cap product

$$\cap : H^{2n}_{\mathcal{D}}(\mathrm{Spec}\, R \times B_\bullet \mathrm{GL}_N(R), \mathbb{R}(n)) \otimes H_i(B_\bullet \mathrm{GL}_N(R), \mathbb{R}) \to H^{2n-i}_{\mathcal{D}}(\mathrm{Spec}\, R, \mathbb{R}(n))\,.$$

The map $H_j(\mathrm{GL}(R), \mathbb{R}) \to H^{2m-j}_{\mathcal{D}}(R, \mathbb{R}(m))$ we wanted is defined to be $\mathrm{ev}^*(c_m) \cap -$. Not very enlightening, perhaps! In fact, making the Beilinson regulator "explicit" in various contexts is an important research problem that people work on.

4. L-functions

Remark 4.1. Recall, if F is a number field which is Galois over $\mathbb{Q}$, and $\mathfrak{p} \subset \mathcal{O}_F$ is a prime ideal lying over a prime $p \in \mathbb{Z}$, then the decomposition group of $\mathfrak{p}$ in $\mathrm{Gal}(F/\mathbb{Q})$ is the sugbroup

$$D_{\mathfrak{p}} := \{\sigma \in \mathrm{Gal}(F/\mathbb{Q}) \,|\, \sigma(\mathfrak{p}) = \mathfrak{p}\}\,.$$

It sits in a short exact sequence

$$1 \to I_{\mathfrak{p}} \to D_{\mathfrak{p}} \to \mathrm{Gal}(\mathbb{F}_{\mathfrak{p}}/\mathbb{F}_p) \to 1$$

where $\mathbb{F}_{\mathfrak{p}} := \mathcal{O}_F/\mathfrak{p}$ is the residue field at $\mathfrak{p}$. The kernel $I_{\mathfrak{p}}$ is called the inertia group of $\mathfrak{p}$. Passing to the inverse limit, we get a sequence

$$1 \to I_p \to D_p \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to 1$$

and call $D_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the decomposition group of p.

For simplicity, suppose that X is a smooth projective variety defined over $\mathbb{Q}$. This is no real restriction because whenever we consider a smooth projective X defined over a number field F, we may consider it as defined over $\mathbb{Q}$ via

$$X \to \mathrm{Spec}\, F \to \mathrm{Spec}\, \mathbb{Q}\,.$$

Fix an $i \in \{0, \dots, 2\dim X\}$. For a prime number p, let $\mathrm{Frob}_p \in D_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denote a lifting of the Frobenius element in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ (so Frob_p is only unique up to conjugation by elements of the inertia group I_p).

Definition 4.2. The Euler factor of $H^i(X)$ at p is

$$P_p(H^i(X), T) := \det(1 - \mathrm{Frob}_p^{-1} \cdot T | H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p})$$

for a prime $\ell \neq p$.

Remarks 4.3. (1) The notation $H^i(X)$ used here is purely formal. But, for the experts/curious, $H^i(X)$ can be given meaning; it is actually something called a "pure motive".

(2) The superscript I_p means the part of $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ fixed by the inertia action.

(3) If p is a prime of good reduction for X, i.e. there exists a projective model for X whose special fibre is smooth, then it follows from smooth and proper base change that I_p acts trivially on $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.

(4) It is conjectured (by Serre, I think) that $P_p(H^i(X), T)$ does not depend on the choice of prime $\ell \neq p$, and that $P_p(H^i(X), T) \in \mathbb{Z}[T]$. Deligne's proof of the Weil conjectures implies this is true when p is a prime of good reduction for X. We will have to assume this conjecture from here on to even begin (though it is known in some interesting cases).

Definition 4.4. The L-function of $H^i(X)$ is, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, defined to be the Euler product

$$L(H^i(X), s) := \prod_{p \text{ prime}} \frac{1}{P_p(p^{-s})}\,.$$

Note that if one removes the (finitely many) factors for primes of bad reduction, then the product converges absolutely for $\mathrm{Re}(s) > \frac{i}{2} + 1$, again by Deligne's proof of the Weil conjectures.

Examples 4.5. (1) Let $X = \operatorname{Spec} F$ for a number field F. Then one finds that

$$L(H^0(X), s) = \prod_{\mathfrak{p} \subset \mathcal{O}_F \text{ prime ideal}} \frac{1}{(1 - \mathrm{Nm}(\mathfrak{p})^{-s})} = \zeta_F(s)$$

which is precisely the Euler product for the Dedekind zeta function $\zeta_F(s)$.

(2) Let $X = E$ be an elliptic curve over $\mathbb{Q}$. Then

$$P_p(H^1(X), T) = 1 - a_p T + p\epsilon(p) T^2$$

where $a_p := 1 + p - |X(\mathbb{F}_p)|$ and

$$\epsilon(p) = \begin{cases} 1 & \text{if } p \text{ is a good reduction prime for } X \\ 0 & \text{otherwise}\,. \end{cases}$$